COMPARATIVE ANALYSIS OF OPTIMAL PORTFOLIOS: A MATHEMATICAL AND EMPIRICAL REASSESSMENT OF THE MARKOWITZ AND SHARPE MODELS

Abdullayeva Dilafruz Sharofiddin qizi
National University of Uzbekistan named after Mirzo Ulugbek usmonovadilafruz37@gmail.com
Usmonov Fakhriddin Sharofiddin o'g'li
Tashkent University of Information Technologies named after Muhammad al–Khwarizmi usmonovfaxriddin77@gmail.com
Usmonov Fazliddin Sharofiddin o'g'li
"UNICON.UZ" LLC gentlemanfu@gmail.com

Abstract: *The evolution of Modern Portfolio Theory (MPT) is fundamentally defined by the tension between theoretical completeness and practical implementability. This research provides a rigorous comparative analysis of the two primary paradigms in asset allocation: the Markowitz Mean-Variance framework and the Sharpe Single-Index Model (SIM). While the Markowitz approach offers a theoretically complete representation of the opportunity set through the full variance-covariance matrix, its practical application is frequently hindered by the "curse of dimensionality". We examine how the quadrating expanding parameter space of the Markowitz model exacerbates estimation risk, transforming the framework into an "error maximizer". Conversely, the Sharpe SIM provides computational parsimony and structural stability by imposing a diagonal residual covariance structure, effectively serving as a form of implicit regularization. This article delineates the structural trade-offs between estimation accuracy (variance) and specification error (bias).*

Keywords: *Markowitz Mean-Variance Optimization, Sharpe Single-Index Model, Estimation Risk, Efficient Frontier, Bias-Variance Trade-off.*

INTRODUCTION

The formalization of portfolio selection by Harry Markowitz transformed investment management into a rigorous constrained optimization problem. The Markowitz Mean-Variance Model evaluates portfolios based on the trade-off between expected return and risk, where risk is measured by portfolio variance.

However, the implementation of the Markowitz framework faces a significant statistical barrier known as the curse of dimensionality. For an investment universe of $\mathbf{N}$ risky assets, a mean-variance optimizer requires the estimation of $\mathbf{N}$ expected returns and $\frac{\mathbf{N(N+1)}}{\mathbf{2}}$ unique entries in the covariance matrix $\boldsymbol{\Sigma}$.

$$\text{Number of expected return estimates} = \mathbf{N}$$

$$\text{Number of unique covariance matrix elements} = \frac{\mathbf{N(N+1)}}{\mathbf{2}}$$

Therefore, the total number of parameters required by the Markowitz model is:

$$\mathbf{N} + \frac{\mathbf{N(N+1)}}{\mathbf{2}}$$

For example, if the investment universe consists of **500** assets, the number of unique covariance matrix elements is:

$$\frac{\mathbf{500(500+1)}}{\mathbf{2}} = \mathbf{125250}$$

This means that more than 125,000 covariance-related parameters must be estimated, which often exceeds the amount of reliable historical data available. As a result, the optimizer may become highly unstable and may overweight assets with positive return estimation errors or underestimated risks. This issue is commonly referred to as the error maximization problem.

William Sharpe addressed this limitation by proposing the Single-Index Model, which reduces the dimensionality of the portfolio optimization problem. Instead of estimating all pairwise covariances between assets, the Sharpe model assumes that asset returns are primarily related to a single market factor. This reduces the required parameter space from $\mathbf{O(N^2)}$ to $\mathbf{O(N)}$, making the model more practical for large investment universes.

5.13 Mathematical Foundations of the Markowitz Mean-Variance Model

The Markowitz model assumes an investment universe consisting of **N** risky assets. These assets are represented by a random return vector **R**, a mean return vector **μ**, and a positive-definite covariance matrix **Σ**.

$$\mathbf{R} = \begin{bmatrix} \mathbf{R_1} \\ \mathbf{R_2} \\ \vdots \\ \mathbf{R_N} \end{bmatrix}$$

$$\boldsymbol{\mu} = \mathbb{E}[\mathbf{R}] = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_N \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \mathbf{Cov(R)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix}$$

The covariance matrix is assumed to be positive definite:

$$\boldsymbol{\Sigma} \succ \mathbf{0}$$

This condition ensures that portfolio variance is strictly positive for any non-zero portfolio weight vector.

5.13.1 Portfolio Return and Portfolio Variance

Let **w** denote the portfolio weight vector:

$$\mathbf{w} = \begin{bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \\ \vdots \\ \mathbf{w_N} \end{bmatrix}$$

The expected return of the portfolio is given by:

$$\boldsymbol{\mu}_p = \mathbf{w}^\top \boldsymbol{\mu}$$

The variance of the portfolio is:

$$\sigma_p^2 = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$$

The full-investment constraint requires that the sum of portfolio weights equals one:

$$\mathbf{w}^\top \mathbf{1} = 1$$

where

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

5.13.2 Lagrangian Derivation

The Markowitz optimization problem minimizes portfolio variance subject to a target expected return $\boldsymbol{\mu}^*$ and a full-investment constraint:

$$\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$$

subject to:

$$\mathbf{w}^\top \boldsymbol{\mu} = \boldsymbol{\mu}^*$$
$$\mathbf{w}^\top \mathbf{1} = 1$$

The Lagrangian function is defined as:

$$\mathcal{L}(\mathbf{w}, \lambda, \gamma) = \frac{1}{2} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} - \lambda(\mathbf{w}^\top \boldsymbol{\mu} - \boldsymbol{\mu}^*) - \gamma(\mathbf{w}^\top \mathbf{1} - 1)$$

Taking the first-order derivative with respect to **w** and setting it equal to zero gives:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \boldsymbol{\Sigma} \mathbf{w} - \lambda \boldsymbol{\mu} - \gamma \mathbf{1} = 0$$

Therefore:

$$\boldsymbol{\Sigma} \mathbf{w} = \lambda \boldsymbol{\mu} + \gamma \mathbf{1}$$

Solving for the optimal portfolio weight vector gives:

$$\mathbf{w}^* = \boldsymbol{\Sigma}^{-1}(\lambda \boldsymbol{\mu} + \gamma \mathbf{1})$$

5.13.3 Scalar Constants and Efficient Frontier

The efficient frontier can be expressed using three scalar constants:

$$A = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}$$
$$B = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$
$$C = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}$$

The determinant-like term is:

$$D = AC - B^2$$

The variance of the efficient portfolio with target return $\boldsymbol{\mu}^*$ is:

$$\sigma_p^2 = \frac{C(\boldsymbol{\mu}^*)^2 - 2B\boldsymbol{\mu}^* + A}{D}$$

The expected return of the global minimum-variance portfolio is:

$$\boldsymbol{\mu}_{GMVP} = \frac{B}{C}$$

The variance of the global minimum-variance portfolio is:

$$\sigma^2_{GMVP} = \frac{1}{C}$$

Thus, the Markowitz efficient frontier is a hyperbola in the expected return-standard deviation space.

5.14 Mathematical Foundations of the Sharpe Single-Index Model

The Sharpe Single-Index Model simplifies the covariance structure of asset returns by assuming that returns are mainly driven by one common market factor.

For asset **i**, the return is defined as:

$$\mathbf{R_i = \alpha_i + \beta_i R_m + \varepsilon_i}$$

where $\mathbf{R_i}$ is the return of asset **i**, $\boldsymbol{\alpha_i}$ is the asset-specific intercept, $\boldsymbol{\beta_i}$ is the sensitivity of asset **i** to the market return, $\mathbf{R_m}$ is the return of the market index, and $\boldsymbol{\varepsilon_i}$ is the asset-specific residual term.

5.14.1 Orthogonality Assumptions

The model assumes that the residual term has an expected value of zero:

$$\mathbb{E}[\boldsymbol{\varepsilon_i}] = \mathbf{0}$$

It also assumes that the residual term is uncorrelated with the market return:

$$\mathbf{Cov(\varepsilon_i, R_m) = 0}$$

The most important simplifying assumption is that residuals are mutually uncorrelated across assets:

$$\mathbf{Cov(\varepsilon_i, \varepsilon_j) = 0} \quad \text{for all } \mathbf{i \neq j}$$

This assumption eliminates the need to estimate all pairwise residual covariances between assets.

5.14.2 Expected Return and Variance in the Sharpe Model

Taking the expectation of the return equation gives:

$$\mathbb{E}[\mathbf{R_i}] = \boldsymbol{\alpha_i} + \boldsymbol{\beta_i}\mathbb{E}[\mathbf{R_m}]$$

Therefore:

$$\boldsymbol{\mu_i = \alpha_i + \beta_i \mu_m}$$

where $\boldsymbol{\mu_m}$ is the expected market return.

The variance of asset **i** is:

$$\boldsymbol{\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon_i}^2}$$

The covariance between assets **i** and **j** is:

$$\boldsymbol{\sigma_{ij}} = \mathbf{Cov(R_i, R_j)} = \boldsymbol{\beta_i \beta_j \sigma_m^2} \quad \text{for } \mathbf{i \neq j}$$

5.14.3 Implied Covariance Matrix

Under the Sharpe Single-Index Model, the covariance matrix can be written as:

$$\Sigma_{SIM} = \sigma_m^2 \beta\beta^\top + D$$

where

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix}$$

and D is a diagonal matrix of residual variances:

$$D = \begin{bmatrix} \sigma^2_{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{\varepsilon_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2_{\varepsilon_N} \end{bmatrix}$$

Thus, the Sharpe model replaces the full covariance matrix with a structured covariance matrix based on market sensitivity and residual variance.

5.14.4 Sherman-Morrison Identity

The Sharpe covariance matrix has a diagonal-plus-rank-one structure. This allows efficient matrix inversion using the Sherman-Morrison identity:

$$\Sigma_{SIM}^{-1} = D^{-1} - \frac{D^{-1}\beta\beta^{T}D^{-1}}{\sigma_m^{-2} + \beta^{T}D^{-1}\beta}$$

This formula reduces computational complexity and allows large portfolios to be optimized more efficiently than with the full Markowitz covariance matrix.

5.14.5 Elton-Gruber-Padberg Algorithm

The Elton-Gruber-Padberg algorithm ranks assets using the excess return-to-beta ratio:

$$\Theta_i = \frac{\mu_i - r_f}{\beta_i}$$

where r_f is the risk-free rate.

Assets are included in the optimal risky portfolio if their excess return-to-beta ratio exceeds a cutoff point:

$$\Theta_i > C^*$$

The cutoff value C^* determines which assets should be selected for the optimal portfolio.

5.15 Bias-Variance Trade-off and Estimation Risk

The choice between the Markowitz model and the Sharpe Single-Index Model can be explained through the bias-variance trade-off.

The Markowitz model has low specification bias because it uses the full covariance matrix. However, it has high estimation variance because it requires the estimation of many parameters. In high-dimensional settings, this can lead to unstable portfolio weights and excessive sensitivity to historical data.

The Sharpe model, on the other hand, introduces structural bias by assuming that all asset co-movements are explained by a single market factor. However, this restriction reduces estimation variance and improves portfolio stability.

The structural difference between the two models is mainly determined by the residual covariance terms. If residual covariances are not zero, then the Sharpe model ignores important sources of co-movement among assets.

In the Markowitz model:

$$\mathrm{Cov}(\varepsilon_i, \varepsilon_j) \neq 0$$

In the Sharpe Single-Index Model:

$$\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0 \quad \text{for all } i \neq j$$

Therefore, the Sharpe model can be interpreted as a restricted version of the Markowitz model.

5.16 Empirical Evidence from Global Markets

Empirical studies show that the relative performance of the Markowitz and Sharpe models depends on market structure, sample size, asset universe, and transaction costs.

In developed markets with large amounts of reliable data, the Markowitz model can provide better control over idiosyncratic risk. However, in high-dimensional or limited-data environments, the Sharpe Single-Index Model often produces more stable out-of-sample performance.

The Markowitz model may generate higher theoretical efficiency, but it can also produce extreme portfolio weights and high portfolio turnover. The Sharpe model usually produces more diversified and stable portfolios because it reduces the number of estimated parameters.

A general comparison can be summarized as follows:

Criterion	Markowitz Model	Sharpe Single-Index Model
Covariance structure	Full covariance matrix	Market-factor-based covariance matrix
Number of parameters	High	Low
Computational complexity	High	Low
Estimation risk	High	Lower
Portfolio stability	Often unstable in small samples	More stable
Diversification control	Strong	Moderate
Practical usefulness	Better in data-rich environments	Better in high-dimensional settings

Conditional Model Selection

The choice between the Markowitz and Sharpe models should depend on the econometric and practical characteristics of the investment environment.

First, the T/N ratio is highly important. Here, T represents the number of historical observations and N represents the number of assets. When T/N is low, the sample covariance matrix becomes poorly conditioned, and the Markowitz model becomes unstable.

$$\frac{T}{N}$$

A low value of this ratio indicates that there are too few observations relative to the number of assets. In such cases, the Sharpe Single-Index Model is often preferable because it imposes a stable factor structure.

Second, the market structure must be considered. If asset returns are largely driven by one common market factor, the Sharpe model can provide a reasonable approximation. However, if returns are influenced by multiple sectoral, macroeconomic, or style factors, then multi-factor models may be more appropriate.

Third, transaction costs and portfolio turnover should be considered. Since the Markowitz model is more sensitive to parameter changes, it can generate higher portfolio turnover. The Sharpe model generally provides more stable weights, which can improve net performance after transaction costs.

Conclusion

The Markowitz Mean-Variance Model and the Sharpe Single-Index Model represent two important approaches to portfolio optimization. The Markowitz model is theoretically comprehensive because it uses the full covariance matrix and directly accounts for the relationships among all assets. However, this theoretical completeness comes at the cost of high parameter requirements, estimation risk, and computational complexity.

The Sharpe Single-Index Model simplifies the optimization problem by assuming that asset returns are driven by a single common market factor. This reduces the number of required estimates and improves portfolio stability. Although the model introduces specification bias by ignoring non-market residual correlations, it often performs better in practical settings where data are limited or the number of assets is large.

Therefore, the relationship between the two models can be understood as a trade-off between theoretical accuracy and practical robustness. The Markowitz model is more suitable for data-rich environments, while the Sharpe model is more suitable for high-dimensional investment universes with limited data.

A hybrid approach may be the most effective solution. Factor models can be used to reduce dimensionality and improve covariance estimation, while Markowitz optimization can be applied afterward to refine portfolio weights and manage idiosyncratic risk.

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